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Coalescing Black Hole Solution in the De-Sitter Universe

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A new coalescing black hole solution of Einstein-Maxwell equation in general relativity is given. The new solution is also found to support the "Nernst Theorem" of thermodynamics in the case of black hole. Thus this solution poses to solve an outstanding problem of thermodynamics and black hole physics.

KEY WORDS: black hole solution; thermodynamics; nernst theorem.

1. INTRODUCTION

In Kastor and Transchen (1993) found a multi-black hole solution in the de-Sitter Universe. Their solution is interesting in that it offers a picture of coalescing black holes (Nakao *et al.*, 1995).

In this paper, we consider the well-known Kerr-Newman solution of Einstein-Maxwell equation in the de-Sitter Universe endowed with NUT (magnetic mass) and magnetic monopole parameters. Under some special consideration we obtain a new solution from this solution.

2. NEW SOLUTION

We consider the following solution:

$$ds^2 = \rho^2\big(\Delta_r^{-1} dr^2 + \Delta_\theta^{-1} d\theta^2\big) - \rho^{-2} J^{-2} \Delta_\theta\, a^2 \sin^2\theta \left[dt - \frac{(r^2 + a^2)}{a} d\varphi\right]^2$$
$$- \Delta_r J^{-2} \rho^{-2} \left[dt - \left(\frac{(\ell - a\cos\theta)^2}{a}\right) d\varphi\right]^2 \tag{1}$$

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where

$$\rho^2 = r^2 + (\ell - a\cos\theta)^2$$
$$\Delta_r = (r^2 + a^2 - \ell^2)\left(1 - \frac{1}{3}\wedge r^2\right) - 2Mr + e^2 + g^2$$
$$\Delta_\theta = 1 + \frac{1}{3}\wedge(\ell - a\cos\theta)^2$$
$$J = 1 + \frac{1}{3}\wedge(a^2 - \ell^2)$$

Besides the cosmological parameter Λ, the solution (1) possesses five parameters: M the mass parameter, a the angular momentum per unit mass parameter, ℓ the NUT (magnetic mass), e the electric charge parameter, and g the magnetic monopole parameter. We call the solution a hot NUT-Kerr-Newman-Kasuya solution (H-NUT-KN-K). We term the solution in the de-Sitter spacetime as being hot, since the de-Sitter spacetime has been interpreted as being hot (Gasperini, 1988).

The solution endowed with NUT parameter does not have any direct physical interpretation. To remove the NUT parameter from the solution (1) and make the solution physically reasonable, we use the interpretation of NUT parameter given by Bonner (1969).

The NUT solution is a stationary and axisymmetric solution of Einstein's empty space equation, but it created a lot of problems from the physical interpretation point of view. An ingenious interpretation was given by Misner (1963, 1967). According to Misner's interpretation, an observer in the NUT spacetime moves forward in time only to find himself in his own past. According to Bonnor (1969), this interpretation of Misner is physically puzzling and resembles, to some extent, the world of Dr. WHO.

Bonnor (1969) gave a quite different interpretation of the NUT metric. According to Bonnor, the NUT solution is due to the field of a spherically symmetric mass together with a semi-infinite massless source of angular momentum along the axis of symmetry. The main purpose of Bonnor was to give an interpretation of the NUT parameter. According to Bonnor's interpretation, the NUT parameter arises due to the strength of the physical singularity on $\theta = \pi$. The NUT metric is singular along the axis of symmetry $\theta = 0$ and $\theta = \pi$. The singularity along $\theta = 0$ is reasonable by a co-ordinate transformation (Bonnor, 1969). But this transformation cannot remove the singularity on $\theta = \pi$. Although the singularity on $\theta = \pi$ may have been removed by other transformations, Bonnor assumed that $\theta = \pi$ is a physical singularity representing the source of the field and he, of course, gave justification for his assumption. Here we may note that Misner (1963, 1967) introduced a periodic time co-ordinate to remove the singularity.

Now according to Bonnor's interpretation, if we consider the NUT parameter ℓ, is due to the strength of the physical singularity on $\theta = \pi$ and if we further consider that $\ell = a$, the solution (1) reduces to the form

$$ds^2 = r^2\left(\Delta_r^{-1}dr^2 + d\theta^2\right) - \Delta_r r^{-2}[dt - ad\varphi]^2 \tag{2}$$

where

$$\Delta_r = r^2\left(1 - \frac{1}{3} \wedge r^2\right) - 2Mr + e^2 + g^2$$

That means the angular momentum of the mass M and the angular momentum of massless rod coalesce, the solution (2) arises.

3. PHYSICAL PROPERTIES

The surface gravity of the event horizon κ_+ as well as of the Cauchy horizon κ_- for the black hole given by (2) can be written as

$$\kappa_\pm = -\frac{1}{6}\frac{1}{r_\pm^2 + 2a^2}(r_+ - r_{++})(r_+ - r_-)(r_+ - r_{++})(r_+ - r_{--}) \tag{3}$$

The surface gravity of the cosmological event horizon κ_{++} as well as of the cosmological Cauchy horizon κ_{--} for the black hole (2) will be given by

$$\kappa_{\pm\pm} = \frac{1}{6}\frac{1}{r_{\pm\pm}^2 + 2a^2}(r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}) \tag{4}$$

where r_+, r_{++}, r_- and r_{--} are the four roots of $\Delta_r = 0$ associated with Eq. (2). The surface $r = r_+$ is the event horizon while the surface $r = r_{++}$ is the cosmological event horizon for observers moving on the world lines of constant r_+ and r_{++} respectively. The surface $r = r_-$ is the Cauchy horizon. Passing through this Cauchy horizon, one comes to the ring singularity $r = 0$, on the other side of which there is a Cauchy cosmological horizon at $r = r_{--}$.

The Hawking temperature of the event horizon T_+ as well as of the Cauchy horizon T_- for the black hole given by (2) will be given by

$$T_\pm = \frac{K_\pm}{2\pi k_B} \tag{5}$$

The Hawking temperature of the cosmological event horizon T_{++} as well as of the cosmological Cauchy horizon T_{--} will be given by

$$T_{\pm\pm} = \frac{K_{\pm\pm}}{2\pi k_B} \tag{6}$$

where k_B is the Boltzmann's constant.

The Behenstein-Smarr (BS) differential formulae for the event horizon as well as for the Cauchy horizon can be written as

$$\delta M = \frac{1}{8\pi}L_+\kappa_+\delta A_+ + L_+\Omega_+\delta J \tag{7}$$

and

$$\delta M = \frac{1}{8\pi} L_- \kappa_- \delta A_- + L_- \Omega_- \delta M \tag{8}$$

respectively, where

$$L_\pm = \frac{A_\pm}{A_\pm \mp 4a^2}$$
$$A_\pm = \pm 4\pi (r_\pm^2 + 2a^2)$$
$$\Omega_\pm = \frac{a}{r_\pm^2 + 2a^2}, \quad J = aM$$

A new BS differential formula can be constructed as

$$\delta M = L_+ \left(\frac{1}{16\pi} \kappa_+ \delta A_+ + \Omega_+ \delta J_+ \right) + L_- \left(\frac{1}{16\pi} \kappa_- \delta A_- + \Omega_- \delta J_- \right) \tag{9}$$

where $J_+ = J_- = J/2$.

Equations (7), (8) and (9) are mathematically equivalent. Equation (9) can be cast into the form

$$\delta M = T_+ \delta S_+ + L_+ \Omega_+ \delta J_+ + T_- \delta S_- + L_- \Omega_- \delta J_- \tag{10}$$

where

$$T_\pm = \frac{\kappa_\pm}{2\pi k_B}, \quad S_\pm = \frac{k_B}{8} \left[A_\pm + 4\pi a^2 \ \ln \ \frac{|A_\pm \mp 4a^2|}{4\pi a^2} \right] \tag{11}$$

$T_\pm$ is the temperature and $S_\pm$ is the entropy of the event horizon and Cauchy horizon.

Equations (7), (8) and (10) show that the black hole given by (2) is a complex thermodynamic system composed of two subsystems. Equations (10) and (11) also show that the event horizon and the Cauchy horizon both contribute to the entropy of the black hole (2). The new entropy of the black hole (2) can be written as

$$\begin{aligned} \bar{S} &= S_+ + S_- \\ &= \frac{4\pi M^2 (M^2 + a^2)}{2M^2 + a^2} \end{aligned}$$

Now the BS differential formula in terms of the new entropy can be written as

$$\delta M = \bar{T} \delta \bar{S} + \bar{\Omega} \delta \bar{J} \tag{12}$$

where

$$\bar{T} = \frac{1}{4\pi k_B}\,\frac{M}{M^2 + a^2}$$

$$\bar{\Omega} = \frac{(3M + a^2)a}{(4M^2 + a^2)(M + a^2)}$$

$$\bar{J} = J$$

In case $M \to 0$, $\bar{T} \to 0$ and $\bar{S} \to 0$ in the new defined temperature, the new defined entropy becomes zero. Thus we see that if the mass of the resulting black hole (2) becomes zero, the new temperature as well as the new entropy will be zero. Thus we find the "Nernst Theorem" of thermodynamics which states that the entropy of a system will be zero as its temperature goes to zero holds good. Usually the entropy of a black hole does not become zero as its temperature goes to zero. This fact creates an awkward situation between thermodynamics and black hole physics although there is a striking similarity between the laws of thermodynamics and those of black hole physics.

In the case of cosmological event horizon and cosmological Cauchy horizon, one may expect the same type of result.

4. REMARKS

If the mass of the coalescing black hole given by (2) results into zero, then we find complete analogue between the laws of thermodynamics and those of black hole physics. Thus we find a solution of an outstanding problem of black hole physics and thermodynamics.

Besides the solution of this outstanding problem, we also notice that this work provides a new way to look into the matter of NUT solution.

We started with the solution (1) endowed with NUT parameter. Actually a solution endowed with NUT parameter does not have any direct physical interpretation. But consideration of the NUT parameter as made in this paper gives a new black hole solution which poses to solve an outstanding problem of thermodynamics and black hole physics.

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